Match Is A Derangement Engine

Derangement Probability, Fixed Points, and Dynamic Payout Scaling

Game Architecture Series

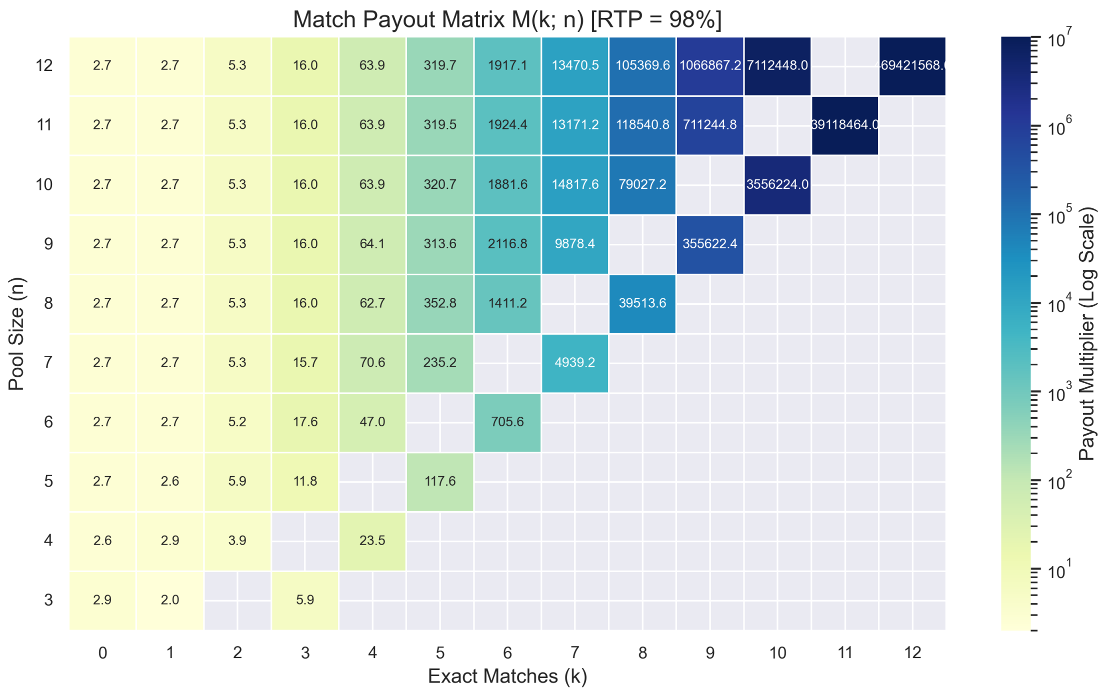

Figure 1: **Exact probability and payout matrix for pool sizes $N \in [3, 12]$.** Moving horizontally increases the target match count k; moving vertically increases the total pool size N. The white diagonal band at $k = N - 1$ represents the mathematical impossibility boundary where $P(N - 1; N) = 0$.

What the Chart Shows

Match is a casino engine built on the combinatorics of random permutations and fixed points. The game setup is simple: a pool of N distinct items is shuffled, and the player wagers on how many items k will land in their exact starting positions.

The surface interface looks like a casual picking or card-sorting game, but underneath lies a rigid probability distribution. The chart above maps the entire operating state space of the game:

- **Horizontal Movement (k):** As the target match count increases, the probability drops sharply, causing payout multipliers to grow exponentially on a logarithmic scale.
- **Vertical Movement (N):** Increasing the pool size expands the domain of possible outcomes and increases the rarity of high-match jackpots.
- **The Impossibility Boundary ($k = N - 1$):** The white stripe running diagonally across the matrix represents an absolute mathematical dead zone. It is physically impossible to match exactly $N - 1$ items out of N.

Game Mechanics

When a pool of N items is shuffled, there are $N!$ total possible outcomes. A match occurs whenever an item's post-shuffle position matches its pre-shuffle index (a mathematical *fixed point*).

The mechanic exhibits three distinct behavioral zones:

1. **Zero Matches ($k = 0$):** Known as a complete *derangement.* As N grows, the probability of zero matches rapidly stabilizes around $1/e \approx 36.7879\%$. No matter how large the pool gets, roughly 36.8% of all random shuffles yield zero matches.

2. **Partial Matches ($1 \leq k \leq N - 2$):** The player selects a subset of k items to stay fixed, while the remaining $N - k$ items must be completely scrambled (deranged).

3. **The Impossibility Wall ($k = N - 1$):** If $N - 1$ items land in their correct slots, the single remaining item is forced into its original slot by default. Thus, exactly $N - 1$ matches can never happen.

Worked Examples: $N = 5$ Pool Size

To understand how the math processes a round, consider a pool size of $N = 5$ items ($5! = 120$ total possible shuffles).

Example 1: Target $k = 0$ (Zero Matches)

The player bets that *no items* will land in their correct positions. The number of complete derangements $D(5)$ is calculated using the inclusion-exclusion formula:

$$D(5) = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 120\left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 44$$

The exact survival probability and corresponding 98% RTP payout multiplier are:

$$P(0; 5) = \frac{44}{120} \approx 0.3667 \quad (36.67\%), \qquad M(0; 5) = \frac{0.98}{0.3667} \approx 2.67\text{x}$$

A 1,000,000-round simulation confirms an empirical survival rate of 36.67%.

Example 2: Target $k = 2$ (Exactly Two Matches)

To get exactly 2 matches out of 5:

1. Choose which 2 items will match: $\binom{5}{2} = 10$ combinations.
2. Derange the remaining $5 - 2 = 3$ items: $D(3) = 3!\left(\frac{1}{2} - \frac{1}{6}\right) = 2$ ways.

Multiplying these steps gives the total winning outcomes $W(2; 5)$ and probability $P(2; 5)$:

$$W(2; 5) = \binom{5}{2} \times D(3) = 10 \times 2 = 20 \text{ ways}$$

$$P(2; 5) = \frac{20}{120} = 0.1667 \quad (16.67\%), \qquad M(2; 5) = \frac{0.98}{0.1667} \approx 5.88\text{x}$$

The General Formula

For any pool size N and target match count $k \in [0, N] \setminus \{N - 1\}$, the exact probability $P(k; N)$ is defined as:

$$P(k; N) = \frac{\binom{N}{k} D(N - k)}{N!} = \frac{D(N - k)}{k!\,(N - k)!}$$

where $D(m)$ represents the subfactorial (derangement) function, computed recursively as:

$$D(0) = 1, \qquad D(1) = 0, \qquad D(m) = (m - 1)\big(D(m - 1) + D(m - 2)\big) \quad \text{for } m \geq 2$$

Pos 1	**Pos 2**	**Pos 3**	**Pos 4**	**Pos 5**
Item A	**Item B**	**Item E**	**Item C**	**Item D**

Figure 2: **Anatomy of a $k = 2$ match outcome on $N = 5$.** Items A and B form fixed points ($k = 2$). Items C, D, and E form a 3-item sub-derangement $D(3) = 2$.

Payout Scaling and Commercial Applications

To guarantee a fixed Return to Player (RTP), payout multipliers $M(k; N)$ scale dynamically based on the target probability:

$$M(k; N) = \begin{cases} \frac{\mathrm{RTP}}{P(k;N)} & \text{if } P(k; N) > 0 \\ 0.00 & \text{if } k = N - 1 \quad \text{(Control Disabled)} \end{cases}$$

Because $P(N; N) = 1/N!$, targeting a perfect match ($k = N$) on larger pool sizes yields massive, lottery-level multipliers (e.g., $N = 10 \implies M = 3,556,224$x) while maintaining the exact global house edge. This structure allows operators to offer configurable, high-volatility jackpot mechanics natively without introducing unhedged financial liability.